SEPARATION OF VARIABLES AND VACUUM STRUCTURE OF $\mathcal{N} = 2$ SUSY QCD

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April 1996

Abstract We show how the method of separation of variables can be used to construct integrable models corresponding to curves describing vacuum structure of four-dimensional $\mathcal{N} = 2$ SUSY Yang-Mills theories. We use this technique to construct models corresponding to $SU(N)$ Yang-Mills theory with $N_f < 2N$ matter hypermultiplets by generalising the periodic Toda lattice. We also show that some special cases of massive $SU(3)$ gauge theory can be equivalently described by the generalisations of the Goryachev-Chaplygin top obtained via separation of variables.

1. In [1], Seiberg and Witten described an intimate relationship between hyperelliptic curves and exact effective actions for $\mathcal{N} = 2$ $SU(2)$ gauge theories in four dimensions. Their results were soon generalised to $SU(N)$ gauge theories [2] and the corresponding curves were found [3]. For the $SU(N)$ gauge theory coupled to $N_f < 2N$ matter hypermultiplets the curves are given by[2]

$$y^2 = \left(x^N - \sum_{i=0}^{N-2} u_i x^i + \frac{\Lambda_{N_f}^{2N-N_f}}{4} \sum_{i=0}^{N_f-N} x^{N_f-N-i} t_i^{(N_f)}(\mathbf{m})\right)^2 - \Lambda_{N_f}^{2N-N_f} \prod_{i=1}^{N_f}(x + m_i), \quad (1)$$

Here u_i, $i = 0, \ldots, N-2$ are gauge-invariant order parameters, m_i, $i = 1, \ldots, N_f$ are masses of the particles and Λ_{N_f} are dynamically generated scales ($\Lambda_{N_f} = \Lambda$ for any $N_f \geq N$). Furthermore for any integer l, $0 \leq l \leq N_f$, $t_k^{(l)}(\mathbf{m})$ are symmetric polynomials in m_i defined by

$$t_k^{(l)}(\mathbf{m}) = \sum_{i_1 < i_2 < \ldots < i_k} m_{i_1} \cdots m_{i_k}, \qquad 1 \leq i_r \leq l. \quad (2)$$

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[2]We use the conventions in which for a sequence $\alpha_k, \alpha_{k+1}, \ldots,$ $\prod_{i=k}^{m} \alpha_i = 1$ and $\sum_{i=k}^{m} \alpha_i = 0$ if $m < k$.

Later it was realised that the Seiberg-Witten theory can be reformulated in terms of classical integrable Hamiltonian systems corresponding to the elliptic Whitham hierarchy [4]. The hyperelliptic curves of $\mathcal{N} = 2$ supersymmetric gauge theories are then interpreted as the spectral curves of the Lax matrix $L(u)$ of the corresponding integrable models. For example in [5] the curve describing $SU(3)$ theory with two massless particles was found to correspond to the Goryachev-Chaplygin top. In general the relevant systems turn out to be the specific limits of the Hitchin models [7] such as elliptic Calogero models [6] or $SL(2)$-spin chains [8]. In the present paper we suggest the use of separation of variables to explicitly construct the required integrable systems. In particular we identify the curves describing $\mathcal{N} = 2$ SUSY QCD with three colours and even number of flavours N_f and pairwise coinciding masses, with separated equations of motion of generalisations of the Goryachev-Chaplygin top constructed recently in [9] via separation of variables. We also propose a generalisation of the periodic Toda lattice whose separated equations of motion describe the vacuum structure of the four-dimensional $\mathcal{N} = 2$ SUSY $SU(N)$ Yang-Mills theory coupled to $N_f < 2N$ massive matter hypermultiplets.

2. We begin by showing how separation of variables can provide a constructive procedure of building integrable models corresponding to complex curves that describe the vacuum structure of $\mathcal{N} = 2$ supersymmetric gauge theories. Assume that we have a classical Hamiltonian system integrable in the Liouville-Arnold sense with n degrees of freedom. Let H_i, $i = 1, ..., n$, be its first integrals in involution. Then, at least locally, there exist canonical variables (p_i, q_i) which allow for separation of the Hamilton-Jacobi equation thus leading to a system of n-equations $\Phi_i(p_i, q_i; H_1, \ldots, H_n) = 0$. Each such equation describes a curve in variables (p_i, q_i). In most cases the functions Φ_i coincide, i.e. $\Phi_i = \Phi$, $i = 1, \ldots, n$, and thus the separated equations describe identical curves. The problem of solving the system reduces to solving the equation $\Phi(p_i, q_i; H_1, \ldots, H_n) = 0$. Having solved the integrable system by separation of variables one can construct new integrable systems by adding new terms to Φ. One starts with the separation coordinates (p_i, q_i), and considers a system described by separated equations $\Phi(p_i, q_i; H_1, \ldots, H_n) + \Psi(p_i, q_i) = 0$, where Ψ is an arbitrary function which does not depend on $H_1, \ldots, H_n$. One then writes $H_1, \ldots, H_n$ in terms of original dynamical variables in which the system was defined in the first place and thus obtains a new integrable system which generalises the one one has started with. In this way one can obtain a hierarchy or a family of integrable models which separate in the same coordinates. Although the function Ψ may be arbitrary it is not always easy to find functions that lead to physically interesting models. This method of constructing new integrable models via

separation of variables was considered in [10] in the case of Neumann model and is fully developed in [9].

From the point of view of $\mathcal{N} = 2$ supersymmetric gauge theories, one views p_i, q_i (or some functions of p_i and q_i) as complex variables, and the equation $\Phi(p_i, q_i; H_1, \ldots, H_n) + \Psi(p_i, q_i) = 0$ as a definition of a complex curve. One can then adjust function Ψ in such a way that the resulting curve corresponds to a given gauge field theory and then, by converting to original variables, one can find corresponding integrable dynamical system. For example, as will be shown in next two sections, knowing the curve and the model corresponding to the pure gauge theory, one can construct a model corresponding to the gauge theory coupled to matter.

Let us note that the above interpretation of hyperelliptic curves of $\mathcal{N} = 2$ SUSY gauge theories in terms of separated Hamilton-Jacobi equations agrees with that of [4] in which the curves are identified with spectral curves of the Lax operator $L(u)$. In the modern approach to separation of variables [11] via functional Bethe Ansatz one starts with the eigenvalue problem of the Lax operator, $L(u)\Omega(u) = p(u)\Omega(u)$. For many models, the poles q_i of the Baker-Akhiezer function $\Omega(u)$ Poisson commute with each other and together with the corresponding eigenvalues $p_i = p(q_i)$ (or some functions of p_i) provide the set of separation variables. It is clear that since for each i, p_i is an eigenvalue of $L(q_i)$, (p_i, q_i) lie on the spectral curve of the Lax operator, i.e. $\Phi(p_i, q_i) \equiv \det(p_i - L(q_i)) = 0$. This last equation gives the separated equations of motion since the coefficients of the characteristic polynomial of $L(q_i)$ depend only on the Hamiltonians $H_1, \ldots, H_n$. Thus the spectral curve of $L(u)$ plays the role of a generating function of separated Hamilton-Jacobi equations.

3. As a first illustration of the general procedure of interpreting hyperelliptic curves in terms of separated Hamilton-Jacobi equations of integrable models we consider the $SU(3)$ curves (1) with $N_f = 2n$, $n = 0, 1, 2$, and $m_{i+n} = m_i$, $i = 1, \ldots, n$, and we show that the corresponding integrable system is a generalisation of the Goryachev-Chaplygin top.

The Goryachev-Chaplygin top is constructed from the variables x_i, J_i, $i = 1, 2, 3$ whose Poisson brackets obey the following relations

$$\{J_i, J_j\} = \epsilon_{ijk} J_k, \quad \{J_i, x_j\} = \epsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0.$$

The Hamiltonian of the system is

$$H = \frac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) - bx_1,$$

where b is a free parameter. If the motion of the model is subjected to the constraints

$$x_1^2 + x_2^2 + x_3^2 = 1, \qquad x_1 J_1 + x_2 J_2 + x_3 J_3 = 0,$$

3

then the induced Poison bracket is non-degenerate and there is an additional integral of motion

$$G = 2J_3(J_1^2 + J_2^2) + 2bx_3J_1,$$

thus making the system completely integrable. The Goryachev-Chaplygin top admits the separation coordinates

$$q_1 = J_3 + \sqrt{J^2}, \qquad q_2 = J_3 - \sqrt{J^2},$$

with the conjugate momenta p_1, p_2 given by $\cos(2p_i) = -x_1 + x_3 J_1/q_i$. The separated equations are

$$2b \cos(2p_i) + q_i^2 = 2H + \frac{G}{q_i}, \tag{3}$$

where H and G are now interpreted as separation constants. The left hand side of (3) may be interpreted as new first integrals H_i. Thus equivalently we can define the Goryachev-Chaplygin top by giving the H_1, H_2 as stated.

To recover curves (1) we follow the procedure described above and consider the generalisation of the Goryachev-Chaplygin top given by the separated equations [9]

$$q_i^2 + \frac{1}{4}A_n^2 \sum_{k=0}^{2n-3} q_i^{2n-4-k} t_k^{(2n)}(\mathbf{m}) + A_n q_i^{-1} \prod_{k=1}^{n}(q_i + m_k) \cos(2p_i) = 2H + \frac{G}{q_i}, \tag{4}$$

where $i = 1, 2$. For given n, the generalisation of the Goryachev-Chaplygin top (4) depends on n mass parameters $m_1, \ldots, m_n$ and the scale A_n. The polynomials $t_k^{(2n)}$ are given by (2) with $m_{n+i} = m_i$, $i = 1, \ldots, n$. We recover the original Goryachev-Chaplygin top when $n = 1$, $m_1 = 0$ and $A_1 = 2b$. By setting $x = q_i$ and

$$y = \pm i A_n \prod_{k=1}^{n}(q_i + m_k) \sin(2p_i),$$

we obtain that the equations (4) describe the curves

$$y^2 = \left(x^3 - 2Hx - G + \frac{A_n^2}{4} \sum_{i=0}^{2n-3} x^{2n-3-i} t_i^{(2n)}(\mathbf{m}) \right)^2 - A_n^2 \prod_{i=1}^{2n}(x + m_i),$$

which for $n = 0, 1, 2$ are equivalent to (1) provided we make the identifications:

$$H = \frac{1}{2}u_1, \quad G = u_0, \quad A_0 = \Lambda_0^3, \quad A_1 = \Lambda_2^2, \quad A_2 = \Lambda.$$

We can express H and G in (4) in terms of the natural coordinates J_i, x_i and consider H as a physical Hamiltonian of this new integrable system. Following [9] we define functions

$$\mathcal{J}^{(r)} = \frac{q_1^{r+1} - q_2^{r+1}}{q_1 - q_2}$$

4

for any integer r. In terms of the original variables J_i the functions $\mathcal{J}^{(r)}$ read

$$\mathcal{J}^{(m)} = \sum_{0 \leq k \leq m/2} \binom{m-k}{k} (2J_3)^{m-2k}(J_1^2 + J_2^2)^k, \qquad \mathcal{J}^{(-m)} = (-1)^m (J_1^2 + J_2^2)^{-m+1} \mathcal{J}^{(m-2)},$$

for any non-negative integer m. Using these functions we can write Hamiltonian H of the system described by separated equations (4) as

$$
\begin{aligned}
2H & = J_1^2 + J_2^2 + 4J_3^2 + \frac{A_n^2}{4} \sum_{k=0}^{2n-3} \mathcal{J}^{(2n-k-4)} t_k^{(2n)}(\mathbf{m}) \\
& \quad + A_n \sum_{k=0}^{n} (-\mathcal{J}^{(n-k-1)} x_1 + \mathcal{J}^{(n-k-2)} J_1 x_3) t_k^{(n)}(\mathbf{m}).
\end{aligned}
$$

The other constant of motion is

$$
\begin{aligned}
G & = (2J_3 + \frac{A_n^2}{4} \sum_{k=0}^{2n-3} \mathcal{J}^{(2n-k-5)} t_k^{(2n)}(\mathbf{m}) \\
& \quad + A_n \sum_{k=0}^{n} (-\mathcal{J}^{(n-k-2)} x_1 + \mathcal{J}^{(n-k-3)} J_1 x_3) t_k^{(n)}(m))(J_1^2 + J_2^2).
\end{aligned}
$$

For given n, the Hamiltonian H and the other integral G have the following structure

$$H = \frac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) + P_n(x_i, J_i) + \frac{A_n}{2} \frac{J_1 x_3}{J_1^2 + J_2^2} \prod_{i=1}^{n} m_i,$$

$$G = 2J_3(J_1^2 + J_2^2) + Q_{n+1}(x_i, J_i) - A_n(x_1 + \frac{J_1 x_3 J_3}{J_1^2 + J_2^2}) \prod_{i=1}^{n} m_i$$

where P_n, Q_n are polynomials of degree at most n. Explicitly for all the cases relevant to the $\mathcal{N} = 2$ supersymmetric $SU(3)$ gauge theory the polynomials P_n, Q_n read

$$P_0 = 0, \qquad P_1 = -A_1 \frac{x_1}{2}, \qquad P_2 = \frac{A_2}{2}(J_1 x_3 - 2J_3 x_1) - \frac{A_2(m_1 + m_2)}{2} x_1 + \frac{A_2^2}{8},$$

$$Q_1 = 0, \qquad Q_2 = A_1 J_1 x_3, \qquad Q_3 = -2A_2 x_1(J_1^2 + J_2^2) + A_2(m_1 + m_2)J_1 x_3 + (m_1 + m_2)\frac{A_2^2}{2}.$$

4. Now we proceed to construct models corresponding to curves that describe the vacuum structure of massive $SU(N)$ theories with arbitrary $N_f < 2N$, as given by (1). It is known that for $N_f = 0$ the corresponding model is the periodic Toda lattice. Therefore we seek suitable generalisations of the Toda lattice whose separated equations of motion could lead to all the curves given by (1).

Recall that the periodic Toda lattice is given by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} \pi_i^2 + \sum_{i=1}^{N} e^{x_i - x_{i+1}},$$

where (π_i, x_i), $i = 1, \ldots, N$ are canonical variables and $x_{N+1} = x_1$. Without the loss of generality we can assume that the total momentum $\sum_i \pi_i$, which is a constant of motion, vanishes. In [12] the separation coordinates $q_1, \ldots, q_{N-1}$ for the Toda lattice were constructed as eigenvalues of the matrix L_1 obtained from the $N \times N$ Lax-matrix

$$
L = \begin{pmatrix}
\pi_1 & e^{\frac{1}{2}(x_1-x_2)} & 0 & \ldots & 0 & 0 & e^{\frac{1}{2}(x_N-x_1)} \\
e^{\frac{1}{2}(x_1-x_2)} & \pi_2 & e^{\frac{1}{2}(x_2-x_3)} & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & e^{\frac{1}{2}(x_{N-2}-x_{N-1})} & \pi_{N-1} & e^{\frac{1}{2}(x_{N-1}-x_N)} \\
e^{\frac{1}{2}(x_N-x_1)} & 0 & 0 & \ldots & 0 & e^{\frac{1}{2}(x_{N-1}-x_N)} & \pi_N
\end{pmatrix}
$$

by deleting first row and column. The separated equations are

$$
2\cosh(p_i) - q_i^N = -\sum_{k=0}^{N-2} u_k q_i^k, \tag{5}
$$

$i = 1, \ldots, N-1$. Here $u_0, \ldots u_{N-3}, u_{N-2} = H$ are constants of motion and p_i are momenta conjugate to q_i given by $p_i = \log|e^{x_1-x_2}\det(q_i - L_2)|$, where L_2 is the matrix obtained from L by deleting first two rows and columns. We remark in passing that equations (5) are equivalent to the equation $\det(v - L(u)) = 0$ for the spectral parameter dependent $N \times N$ Lax matrix $L(u)$ (see e.g. [8]) provided we put $v = q_i$, $u = e^{p_i}$.

Following the general procedure described in Section 2 we consider an integrable system constructed from the canonical variables p_i, q_i, $i = 1, \ldots, N-1$ defined above and the separated equations

$$
\frac{1}{2}(P_{n_1}(q_i)e^{p_i} + Q_{n_2}(q_i)e^{-p_i}) - (q_i^N + \frac{\Lambda_{N_f}^{2N-N_f}}{4}\sum_{k=0}^{N_f-N} q_i^{N_f-N-k}t_k^{(N_f)}(\mathbf{m})) = -\sum_{k=0}^{N-2} u_k q_i^k, \tag{6}
$$

$i = 1, \ldots, N-1$. Here P_{n_1}, Q_{n_2} are polynomials such that $P_{n_1}(x)Q_{n_2}(x) = \Lambda_{N_f}^{2N-N_f}\prod_{k=1}^{N_f}(x + m_k)$, and $u_0, \ldots, u_{N-2}$ are first integrals of motion (or, equivalently, separation constants).

The corresponding hyperelliptic curve is obtained by setting

$$
y = \pm\frac{1}{2}(P_{n_1}(q_i)e^{p_i} - Q_{n_2}(q_i)e^{-p_i}), \qquad x = q_i, \tag{7}
$$

and is identical with (1).

Therefore any classical integrable Hamiltonian system given by separated equations (6) provides a generalisation of the Toda lattice which describes a vacuum structure of general $\mathcal{N} = 2$ SUSY QCD. Now we would like to study the structure of this model from the Hamiltonian point of view. In particular we would like to express the model in terms of the natural variables π_i, x_i. Following (5) we choose $u_{N-2} = H$ to be the Hamiltonian of the

system. Using properties of the Vandermonde determinant one easily finds the explicit form of the Hamiltonian

$$H = \sum_{i=1}^{N-1} \frac{-\frac{1}{2}(P_{n_1}(q_i)e^{p_i} + Q_{n_2}(q_i)e^{-p_i}) + q_i^N}{\prod_{j\neq i}(q_i - q_j)}$$
$$+ \frac{\Lambda_{N_f}^{2N-N_f}}{4}(\delta_{N_f,2(N-1)} + \delta_{N_f,2N-1}(\sum_{i=1}^{N-1} q_i + \sum_{k=1}^{N_f} m_k)). \tag{8}$$

Although the explicit expressions for q_i in terms of π_i and x_i cannot be obtained the expressions of the form

$$\sum_{i=1}^{N-1} F(q_i) \prod_{j\neq i}(q_i - q_j)^{-1},$$

where F is any rational function can be found in terms of original variables since they involve only symmetric polynomials $t_k^{(N-1)}(\mathbf{q})$ in the eigenvalues $q_1, \ldots q_{N-1}$ of matrix L_1. The polynomials $t_k^{(N-1)}(\mathbf{q})$ can be easily read off from the characteristic polynomial of L_1, e.g. $t_1^{(N-1)}(\mathbf{q}) = \mathrm{tr} L_1$ and $t_{N-1}^{(N-1)}(\mathbf{q}) = \det L_1$ etc. This suffices to find H as a function of x_i, π_i since, by definition of the p_i,

$$P_{n_1}(q_i)e^{p_i} + Q_{n_2}(q_i)e^{-p_i} = P_{n_1}(q_i)e^{x_1-x_2}|\det(q_i - L_2)| + \frac{Q_{n_2}(q_i)e^{x_2-x_1}}{|\det(q_i - L_2)|}$$

is a rational function of q_i. Furthermore, with no loss of generality we can assume that $n_1 \geq n_2$ and take

$$P_{n_1}(q) = \Lambda_{N_f}^{N-n_1} \prod_{k=1}^{n_1}(q + m_k), \qquad Q_{n_2}(q) = \Lambda_{N_f}^{N-n_2} \prod_{k=n_1+1}^{N_f}(q + m_k).$$

Then, the Hamiltonian H (8) has the following form

$$H = \frac{1}{2}\sum_{i=1}^{N} \pi_i^2 + \sum_{i=2}^{N-1} e^{x_i-x_{i+1}} + \frac{\Lambda_{N_f}^{2N-N_f}}{4}(\delta_{N_f,2(N-1)} + \delta_{N_f,2N-1}(\sum_{k=1}^{N_f} m_k - \pi_1))$$
$$+ \frac{\Lambda_{N_f}^{N-n_1}}{2}(\prod_{k=1}^{n_1}(\pi_2 + m_k)e^{x_1-x_2} + \Lambda_{N_f}^{n_1-n_2} \prod_{k=n_1+1}^{N_f}(\pi_N + m_k)e^{x_N-x_1} + G_{N_f}^{(N)}),$$

where $G_{N_f}^{(N)}$ are polynomials in $\pi_2, \ldots, \pi_N$ of degree $n_1 - 2$. For a given N, $G_{N_f}^{(N)}$ are defined for any $N_f = 0, \ldots, 2N-1$ and are obtained as contributions to H coming from the terms of the form

$$\sum_{i=1}^{N-1} \frac{q_i^k e^{p_i}}{\prod_{j\neq i}(q_i - q_j)} + \pi_2^k e^{x_1-x_2}, \qquad \sum_{i=1}^{N-1} \frac{q_i^k e^{-p_i}}{\prod_{j\neq i}(q_i - q_j)} + \pi_N^k e^{x_N-x_1}.$$

As an explicit example we take n_1, n_2 such that $n_1 - n_2$ is either 0 or 1. The first seven $G_{N_f}^{(N)}$ come out as:

$$G_0^{(N)} = G_1^{(N)} = G_2^{(N)} = 0, \qquad G_3^{(N)} = e^{x_1 - x_3}, \qquad G_4^{(N)} = e^{x_{N-1} - x_1} + e^{x_1 - x_3},$$

$$G_5^{(N)} = \Lambda_5 e^{x_{N-1} - x_1} + (2\pi_2 + \pi_3 + \sum_{k=1}^{3} m_k) e^{x_1 - x_3},$$

$$G_6^{(N)} = (2\pi_N + \pi_{N-1} + \sum_{k=4}^{6} m_k) e^{x_{N-1} - x_1} + (2\pi_2 + \pi_3 + \sum_{k=1}^{3} m_k) e^{x_1 - x_3},$$

and

$$\begin{aligned}
G_7^{(N)} &= \Lambda_7 (2\pi_N + \pi_{N-1} + \sum_{k=5}^{7} m_k) e^{x_{N-1} - x_1} + (\sum_{k<l}^{4} m_k m_l + (2\pi_2 + \pi_3) \sum_{k=1}^{4} m_k \\
&\quad + 3\pi_2^2 + 2\pi_2 \pi_3 + \pi_3^2) e^{x_1 - x_3} + e^{x_1 + x_2 - 2x_3} + e^{x_1 - x_4}
\end{aligned}$$

The $G_{N_f}^{(N)}$ functions listed above suffice to describe all the Hamiltonians of integrable models corresponding to $SU(3)$ and $SU(4)$ gauge theories.

Notice that the generalised Goryachev-Chaplygin Hamiltonian derived in Section 3 can be viewed as a special case of the $N = 3$ Hamiltonian (8) provided we make suitable identifications of q_i, p_i and P_{n_1}, Q_{n_2}.

We would like to conclude the paper by indicating the possibility of yet another description of the curve (1) which makes explicit use of interpretation of y and x as functions on the phase space of the integrable Hamiltonian system given by (6). The Hamiltonians u_k give rise to Hamiltonian vector fields parametrised by the 'times' $t_0, \ldots, t_{N-2}$ and, for any function f on the phase space, given by $\frac{\partial}{\partial t_k} f = \{u_k, f\}$. Using definition (7) of y and x and separated equations (6) we can express the curve (1) as

$$y = \pm \sum_{k=0}^{N-2} x^k \frac{\partial x}{\partial t_k}.$$

We think that this simple equation can prove useful in analysis and interpretation of the canonical form on the hyperelliptic curve, which constitutes the second half of the Seiberg-Witten theory. The analysis of the canonical one-form from this point of view is currently being carried out and we hope to present the results of this investigation soon.

References

[1] N. Seiberg and E. Witten. *Nucl. Phys.* **B426** (1994) 19.

    N. Seiberg and E. Witten. *Nucl. Phys.* **B431** (1994) 484.

[2] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz. *Phys. Lett.* **B344** (1995) 169.
P.C. Argyres and A.E. Faraggi. *Phys. Rev. Lett.* **73** (1995) 3931.

[3] A. Hanany and Y. Oz. *Nucl. Phys.* **B452** (1995) 283.
P.C. Argyres and A.D. Shapere. *Nucl. Phys.* **B461** (1996) 437.

[4] A. Gorsky, I. Krichever, A.Marshakov, A. Mironov and A. Morozov. *Phys. Lett.* **B355** (1995) 466.

[5] A. Marshakov. Exact solutions to quantum field theories and integrable equations. *E-print* `hep-th/9602005`, 1996.
C. Ahn and S. Nam. Integrable structures in supersymmetric gauge theories with massive hypermultiplets. *E-print* `hep-th/9603028`, 1996.

[6] E.J. Martinec. *Phys. Lett.* **B367** (1996) 91.

[7] R. Donagi and E. Witten. *Nucl. Phys.* **B460** (1996) 299.

[8] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov. $\mathcal{N} = 2$ Supersymmetric QCD and Integrable Spin Chains: Rational Case $N_f < 2N_c$. *E-print* `hep-th/963140`.
A. Gorsky, A. Marshakov, A. Mironov and A. Morozov. A Note on Spectral Curve for the Periodic Homogenous XYZ-Spin Chain. *E-print* `hep-th/9604078`.

[9] D.R.D. Scott. *Separation of Variables and Integrability.* University of Cambridge, PhD thesis. 1995.

[10] A.J. Macfarlane. Integrable generalisations of quantum mechanical models integrable via separation of variables. *DAMTP preprint*, 1992.

[11] E.K. Sklyanin. *Progr. Theor. Phys. Suppl.* **118** (1995) 35.

[12] M. Kac and P. van Moerbeke. *Proc. Nat. Acad. Sci. USA* **72** (1975) 1627.
M. Kac and P. van Moerbeke. *Proc. Nat. Acad. Sci. USA* **72** (1975) 2879. P. van Moerbeke. *Invent. Math.* **37** (1976) 45.